Sydney Girls High School
2017
Trial Higher School Certificate
Examination

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- In Questions 11- 14, show all relevant mathematical reasoning and/or calculations
- A mathematics exam reference sheet is provided.

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2017 HSC Examination Paper in this subject.

Total marks – 70

SECTION 1 –

10 marks

- Attempt questions 1 – 10
- Answer on the Multiple Choice sheet provided
- Allow about 15 minutes for this section

SECTION II –

60 marks

- Attempt questions 11 – 14
- Answer on the blank paper provided
- Allow about 1 hours 45 minutes for this section

Name: ______________________________

Teacher: ____________________________

Section I - Total Marks 10

Attempt Questions 1 – 10
Allow about 15 minutes for this section.
Use the multiple-choice answer sheet for Questions 1-10

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1. Which sum is equal to $\sum_{k=1}^{15}(3k-1)$?

   (A) 1 + 2 + 3 + 4+ ………. + 15

   (B) 1 + 4 + 7 + 10+……… + 44

   (C) 2 + 5 + 8 + 11 + ………+ 44

   (D) 2 + 5 + 8 + 11 + ………+ 15

2. What is the remainder when $3x^3+5x^2-4x+3$ is divided by $x+3$?

   (A) -21

   (B) 117

   (C) $3x^2-4x+8$

   (D) $-3x^2+4x-8$

3. Which expression is equivalent to $\cos 3x\ \cos 7x-\sin 3x\ \sin 7x$?

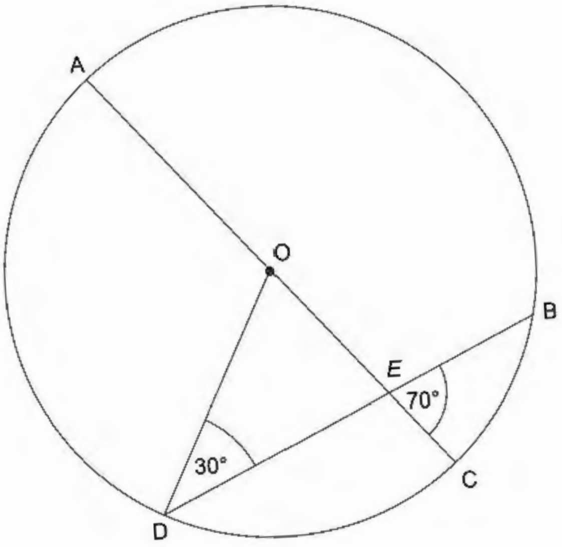
   (A) $\cos 10x-\sin 10x$

   (B) $\cos 4x-\sin 4x$

   (C) $\cos 10x$

   (D) $\cos 4x$

4. In the diagram below, O is the centre of the circle $ABCD$, E is the point of intersection of AC and BD, $\angle ODE = 30^\circ$ and $\angle BEC = 70^\circ$.



What is the size of the angle CAB?

(A) 20°

(B) 30°

(C) 40°

(D) 60°

5. Which expression is equal to $\int \cos^2 \frac{x}{2} dx$

(A) $\frac{1}{2}(x+\sin x)+c$

(B) $\frac{1}{2}(x-\sin x)+c$

(C) $2\cos^3 \frac{x}{2}+c$

(D) $2\sin^3 \frac{x}{2}+c$

6. What is the general solution of the equation $4\sin x + 2\sin x\cos x - \cos x = 2$?

(A) $n\pi - (-1)^n \frac{\pi}{6}$

(B) $n\pi + (-1)^n \frac{\pi}{6}$

(C) $n\pi - (-1)^n \frac{\pi}{3}$

(D) $n\pi + (-1)^n \frac{\pi}{3}$

7. The displacement x of a particle at time t is given by

$$x = 3\sin 2t + 4\cos 2t \ .$$

What is the maximum acceleration of the particle?

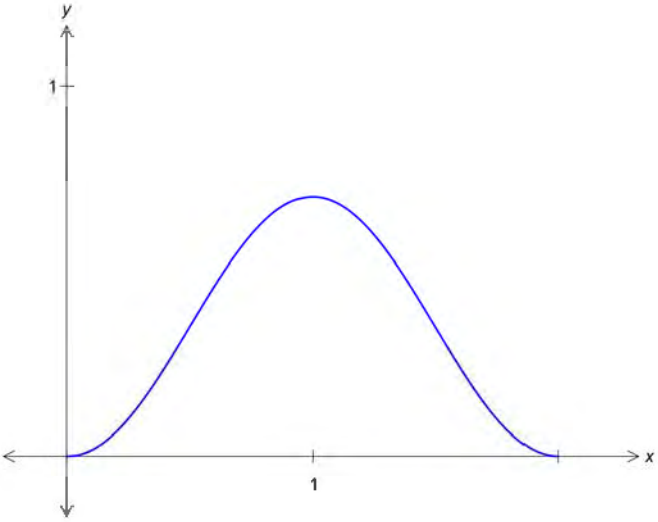
(A) 4

(B) 28

(C) 16

(D) 20

8. The diagram below shows the graph of $y = f(x)$.



Which of the following is a correct statement?

(A) $f''(1) < f(1) < 1 < f'(1)$

(B) $f''(1) < f'(1) < f(1) < 1$

(C) $f(1) < 1 < f'(1) < f''(1)$

(D) $f'(1) < f(1) < 1 < f''(1)$

9. A committee of 7 people is to be formed from a group of 20 students. Among the 20 students are 9 females. What is the number of possible committees containing at least 3 students who are female?

(A) 56 400

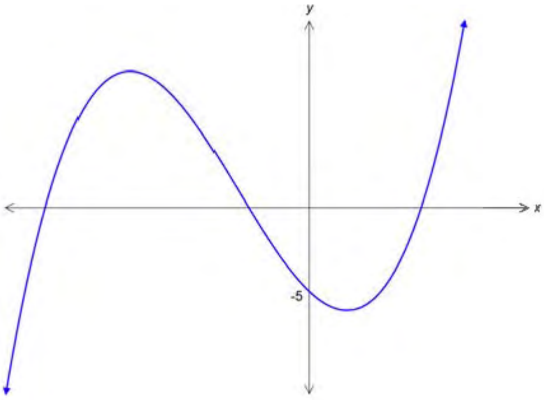
(B) 77 520

(C) 199 920

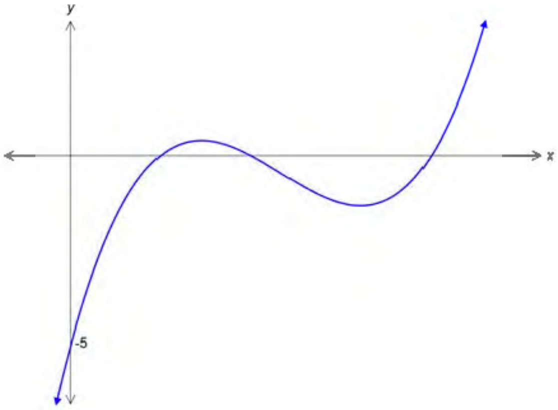
(D) 27 720

10. Consider the polynomial $p(x) = ax^3 + bx^2 + cx - 5$ where a and b are both negative. Which graph below could represent $y = p(x)$?

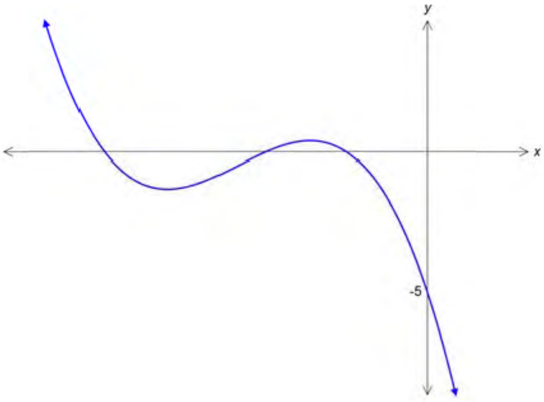
(A)



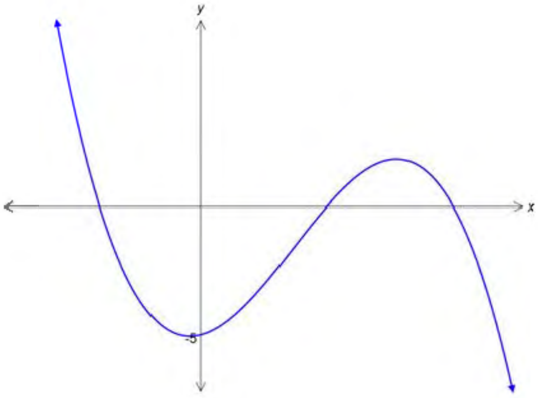
(B)



(C)



(D)



Section II -

60 Marks
Attempt Questions 11 – 14
Allow about 1 hour 45 minutes for this section
Start each question on a new page
In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new page

(a) Find the inverse function of the function $y = x^5 + 3$. 2

(b) Use the substitution $u = x + 3$ to find $\int \frac{x}{\sqrt{x+3}}\, dx$. 3

(c) Differentiate $4\sin^{-1}(3x)$. 2

(d) Evaluate $\lim\limits_{x \to 0} \left(\frac{\cos x \sin x}{2x} \right)$. 2

(e) Solve for : $x + \frac{2}{x+3} < 0$. 3

(f) The letters of the word C R I S I S are arranged randomly to form a six-letter arrangement.

(i) How many different ordered arrangements are possible ? 1

(ii) Find the probability that the ordered arrangement starts and ends with the letter S. 1

(iii) Find the probability that the first letter and the last letter of the ordered arrangement are not the same letter. 1

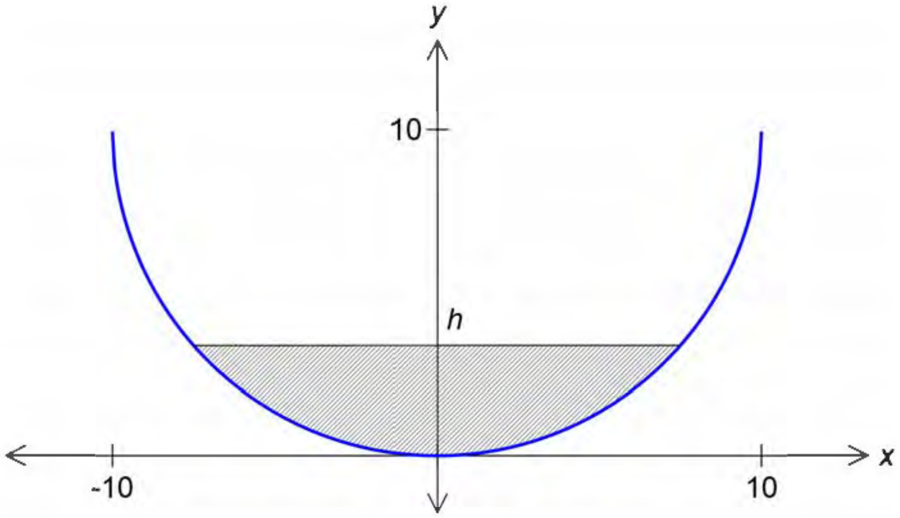
End of Question 11

Question 12 (15 marks) Start a new page

(a) For what values of k is the line $y = 29x + k$ a tangent to the curve $y = x^3 + 2x$? 3

(b) The acute angle between the lines $y = 2x - 3$ and $y = mx + 1$ is $60°$. Find the two possible values of m . 3

(c)



The shaded area shown above is bounded by the curve $y = 10 - \sqrt{100 - x^2}$ and the line $y = h$.

(i) Show that the volume V formed when the shaded area is rotated around the y axis is given by $V = 10\pi h^2 - \dfrac{\pi h^3}{3}$. 2

(ii) A semicircle is rotated around the y axis to form a hemispherical bowl of radius 10cm. The bowl is filled with water at a constant rate of $5\,\text{cm}^3\,\text{s}^{-1}$. 3 Find the rate at which the water level is rising when the water level is 2cm.

Question 12 (continued)

(d) A chocolate cake which is initially at a temperature of $22°C$, is placed in a refrigerator that has a constant temperature of $2°C$. The cooling rate of the cake is proportional to the difference between the temperature of the refrigerator and the temperature T, of the cake. That is, T satisfies the differential equation

$$\frac{dT}{dt} = -k(T-2)$$

where t is the number of minutes after the cake is placed in the refrigerator.

(i) Show that $T = 2 + Ae^{-kt}$ satisfies the differential equation. 1

(ii) The temperature of the cake is $10°C$ after 15 minutes. Find the temperature of the cake after 20 minutes, giving your answer to the nearest degree. 3

End of Question 12

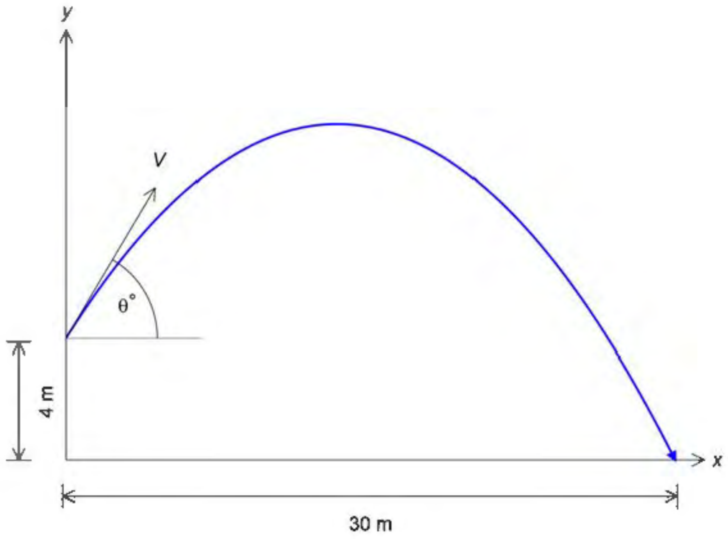
Question 13 (15 marks) Start a new page

(a) The tide can be modelled using simple harmonic motion. At a particular location, the high tide is 10 metres and the low tide is 4 metres. At this location the tide completes 2 full periods every 25 hours. Let x represent the tide level in metres and t be the time in hours after the first low tide today.

(i) If the tide described above can be modelled by the function $x = a + b\cos(nt)$, find the values of a, b and n. 2

(ii) The first low tide is at 3am today. What is the latest time tomorrow at which the tide is increasing at the fastest rate? 2

(b)



A rock is projected with a speed V ms^{-1} from a point 4 metres above a flat sea. The angle of projection to the horizontal is θ, as shown. Assume that the equations representing the acceleration of the rock are $\ddot{x} = 0$ and $\ddot{y} = -10$.

(i) Let (x, y) be the position of the rock at time t seconds after it is projected, and before the rock hits the water.

It is known that $x = Vt\cos\theta$. Show that $y = Vt\sin\theta - 5t^2 + 4$. 2

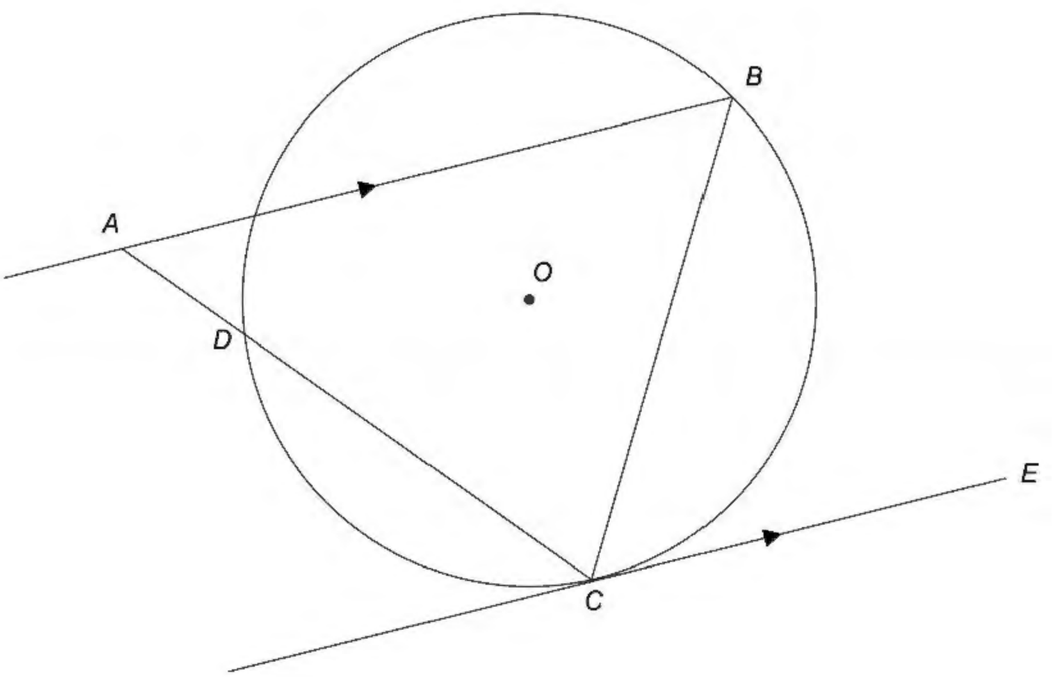
(ii) Suppose the rock hits the water 30 metres away as shown in the diagram.

Find the value of V (correct to 2 decimal places) if $\theta = \tan^{-1}\dfrac{5}{12}$. 3

(iii) For the projection described in part (ii), find the maximum height above sea level that the rock achieved. 2

Question 13 (continued)

(c)



In the diagram above B, C and D are points on the circle with centre at O. The line CE is tangent to the circle at C so that AB is parallel to CE.

(i) Copy the diagram onto your writing paper.

(ii) Show that $\angle CBD = \angle CAB$ 2

(iii) Deduce that $CB^2 = AC \times DC$. 2

End of Question 13

Question 14 (15 marks) Start a new page

(a) Prove by mathematical induction that 3

$$n^3+(n+1)^3+(n+2)^3$$

is divisible by 3 for n = 1, 2, 3, …….

(b) Evaluate $\int_0^{\frac{1}{2}} \cos(2\cos^{-1}x)\,dx$. 2

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are the ends of a focal chord on the parabola $x^2=4ay$.

(i) Show that $pq=-1$. 2

(ii) The locus of the point of intersection of the normals at the ends of the chord PQ is a parabola. Find the focus and directrix of this parabola in terms of a . 4

(d) It is given that $P(x)=(x-a)^3+(x-b)^2$ and the remainder when $P(x)$ is divided by $(x-b)$ is -8. Prove that $P(x)$ has no stationary points. 4

End of paper

Sydney Girls High School

Mathematics Faculty

Multiple Choice Answer Sheet
Trial HSC Mathematics Extension 1

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct

A ~~●~~ B ~~●~~ C ○ D ○

Student Number: ANSWERS

Completely fill the response oval representing the most correct answer.

1. A ○ B ○ C ● D ○
2. A ● B ○ C ○ D ○
3. A ○ B ○ C ● D ○
4. A ● B ○ C ○ D ○
5. A ● B ○ C ○ D ○
6. A ○ B ● C ○ D ○
7. A ○ B ○ C ○ D ●
8. A ○ B ● C ○ D ○
9. A ● B ○ C ○ D ○
10. A ○ B ○ C ● D ○

SOLUTIONS

Trial HSC 2017 (Mathematics Extension 1)

SECTION I

1. $$\sum_{k=1}^{15}(3k-1) = (3-1)+(6-1)+\ldots\ldots+(45-1)$$

$$= 2+5+\ldots\ldots+44$$

(C)

2. $P(x) = 3x^3+5x^2-4x+3$

$P(-3) = 3\times -27 + 5\times 9 + 12 + 3$

$= -21$

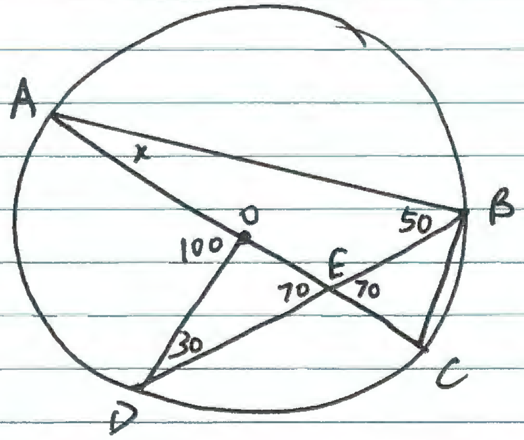
(A)

3. $\cos 3x \cos 7x - \sin 3x \sin 7x = \cos(3x+7x)$

$= \cos(10x)$

(C)

4.



Let $\angle CAB = x$.

$\angle OED = 70^\circ$ (vert. opp. ∠s)

$\angle AOD = 30^\circ + 70^\circ$ (ext. ∠ of Δ)

$= 100^\circ$

$\angle ABD = \frac{100}{2}$ (∠ at circumf. is half ∠ at centre)

$= 50^\circ$

$x + 50 = 70$ (ext. ∠ of Δ)

$x = 20$

(A)

5. $\cos 2x = 2\cos^2 x - 1$

$\cos x = 2\cos^2 \frac{x}{2} - 1$

$$\int \cos^2 \frac{x}{2}\,dx = \frac{1}{2}\int (\cos x + 1)\,dx$$

$$= \frac{1}{2}(\sin x + x) + C$$

(A)

6. $4\sin x + 2\sin x\cos x - \cos x = 2$

$4\sin x - 2 + 2\sin x\cos x - \cos x = 0$

$2(2\sin x - 1) + \cos x(2\sin x - 1) = 0$

$(2 + \cos x)(2\sin x - 1) = 0$

$\cos x = -2$ or $\sin x = \frac{1}{2}$

(No soln.)

General soln. $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$

$\therefore x = n\pi + (-1)^n \times \frac{\pi}{6}$ Ⓑ

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7. $x = 3\sin 2t + 4\cos 2t$

$\dot{x} = 6\cos 2t - 8\sin 2t$

$\ddot{x} = -12\sin 2t - 16\cos 2t$

$= -a\sin(2t + \alpha)$

where $a^2 = 12^2 + 16^2$ $a = 20$

$\ddot{x} = -20\sin(2t + \alpha)$

$\therefore$ Max acceleration is 20. Ⓓ

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8. $f'(1) = 0$ (stat. pt. at $x = 1$) $f''(1) < 0$ (concave down at $x = 1$) $0 < f(1) < 1$

$\therefore f''(1) < f'(1) < f(1) < 1$ Ⓑ

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9. 20 students, 9 female

\# of committees with at least 3 F = Total − \# of committees with less than 3 F

$= {}^{20}C_7 - \left({}^{11}C_7 + {}^{11}C_6 \times {}^{9}C_1 + {}^{11}C_5 \times {}^{9}C_2\right)$

$= 56\,400$ Ⓐ

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10. $p(x) = ax^3 + bx^2 + cx - 5$ $a < 0$ $b < 0$

Since $a > 0$, only graphs C and D are possible.

$p'(x) = 3ax^2 + 2bx + c$

$p''(x) = 6ax + 2b$ $p''(0) = 2b < 0$ since $b < 0$

$\therefore$ Graph is concave down at $x = 0$.

Graph C is the only one satisfying this condition (between C and D). Ⓒ

HSC Extension I (Mathematics Trial)
2017

11

a)

$$y = x^5 + 3$$

$$x^5 = y - 3$$

$$x = (y-3)^{1/5}$$

$$y^{-1} = (x-3)^{1/5}$$

most students got it correct.

b)

$$u = x+3$$

$$x = u-3$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

a

$$\int \frac{x}{x+3}\, dx = \int \frac{u-3}{\sqrt{u}}\, du$$

$$= \int (u^{1/2} - 3u^{-1/2})\, du$$

$$= \frac{u^{3/2}}{3/2} - 3\frac{u^{1/2}}{1/2} + c$$

$$= \frac{2u^{3/2}}{3} - 6u^{1/2} + c$$

$$\int \frac{x}{x+3} = \frac{2(x+3)^{3/2}}{2} - 6\sqrt{x+3} + c$$

Some students did not sub the value of u back in the integral and they have been penalised.

11

c) $y = 4\sin^{-1}(3x)$

$= 4\dfrac{1}{\sqrt{1-9x^2}} \cdot 3$

$= \dfrac{12}{\sqrt{1-9x^2}}$

Some students ~~f~~ failed to simplify the final results, so lost mark on that accound.

d) $\lim\limits_{x\to 0} \dfrac{\cos x \sin x}{2x}$

$= \dfrac{1}{2}\lim\limits_{x\to 0} \dfrac{2\sin x\cos x}{2x}$

$= \dfrac{1}{2}\lim\limits_{x\to 0} \dfrac{\sin 2x}{2x}$

$= 1/2 \times 1$

Some students were confused in this question about the trig involved. Hence they lost mark on account of it.

11
e)

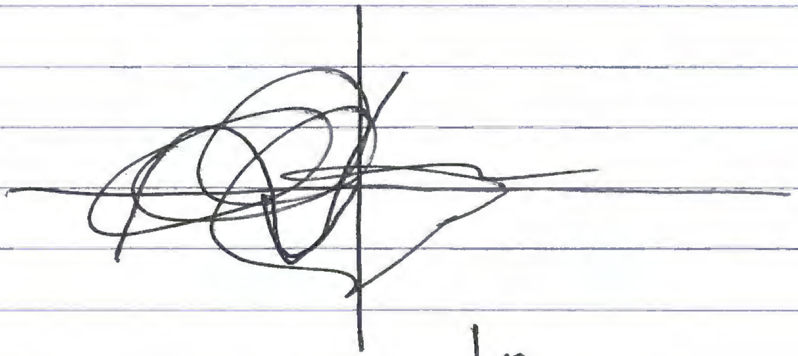
$$x + \frac{2}{x+3} < 0$$

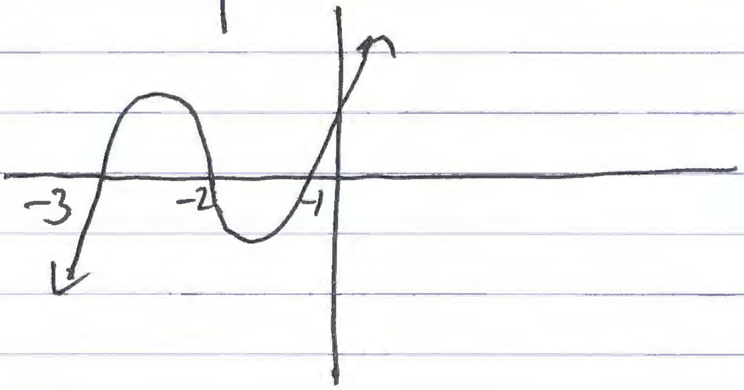
$$(x+3)^2 x + 2(x+3) < 0$$

$$(x+3)[(x+3)x + 2] < 0$$

$$(x+3)(x^2+3x+2) < 0$$

$$(x+3)(x+2)(x+1) < 0$$





$x < -3$, $-2 < x < -1$

Some students failed to provide either correct factorisation or complete solution. Hence marks were deducted.

11

f (i) $$\frac{6!}{2!2!} = 180$$

(ii) no. of arrangements $$\frac{4!}{2!} = 12$$

first and final letter selects themselves automatically

$$\text{Probability} = \frac{12}{180} = \frac{1}{15}$$

(iii) There are two pairs of same letters (I, I and S, S)

$$\text{desired Probability} = 1 - \frac{2}{15} = \frac{13}{15}$$

Most students got part (i) correct. However, in calculating the desired probability in second part, either they did not calculate it or provided the wrong result. So they have been ~~marked~~ penalised.

Q12 Ext 1 Trial 2017

a) $y' = 29$

$y' = 3x^2 + 2$

$29 = 3x^2 + 2$

$x^2 = 9$

$x = \pm 3$

$x = 3 \rightarrow y = 33$

$x = -3 \rightarrow y = -33$

$\therefore k = 33 - 87$ or $k = -33 + 87$

$k = -54$ or 54

b) $\tan\theta = \left|\dfrac{2-m}{1+2m}\right|$

$\sqrt{3} = \dfrac{2-m}{1+2m}$ or $-\sqrt{3} = \dfrac{2-m}{1+2m}$

$\sqrt{3} + 2\sqrt{3}m = 2 - m$

$m(2\sqrt{3}+1) = 2 - \sqrt{3}$

$m = \dfrac{2-\sqrt{3}}{2\sqrt{3}+1}$

$= \dfrac{5\sqrt{3}-8}{11}$

$\doteq 0.06$

or

$-\sqrt{3} - \sqrt{3}(2m) = 2 - m$

$m(1 - 2\sqrt{3}) = 2 + \sqrt{3}$

$m = \dfrac{2+\sqrt{3}}{1-2\sqrt{3}}$

$= \dfrac{-8-5\sqrt{3}}{11}$

$\doteq -1.515$

Some student didn't use the formula correctly.

c) i) $100 - x^2 = (10-y)^2$

$\cancel{100} - x^2 = \cancel{100} - 20y + y^2$

$x^2 = 20y - y^2$

$V = \pi \int_0^h 20y - y^2 \, dy$

$= \pi \left[\dfrac{20y^2}{2} - \dfrac{y^3}{3} \right]_0^h$

$= \pi \left[10h^2 - \dfrac{h^3}{3} \right]$

ii) $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$

$5 = (20\pi h - \pi h^2) \times \dfrac{dh}{dt}$

$\therefore$

$\dfrac{dh}{dt} = \dfrac{5}{40\pi - 4\pi}$

$= \dfrac{5}{36\pi}$ cm/s

Some students used the wrong volume

d i) $T = 2 + Ae^{-kt}$ $\therefore Ae^{-kt} = T - 2$

$\dfrac{dT}{dt} = -kAe^{-kt}$

$= -k(T-2)$

d ii) $A = 20$

$10 = 2 + 20e^{-15k}$

$\therefore k \doteq 0.0611$

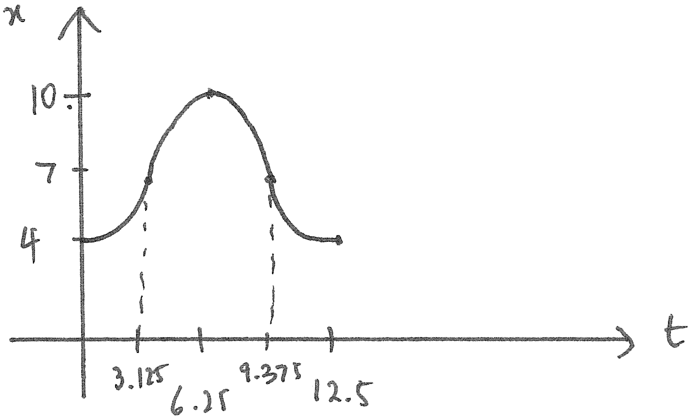
$\therefore T = 2 + 20e^{-20k}$

$= 7.89..$

$= 8°C$

Question 13

(a) (i) $P = \frac{25}{2} = \frac{2\pi}{n}$ $\therefore$ $n = \frac{4\pi}{25}$



$a = \frac{6}{2} = 3$

Centre of motion $= 7$

$$x = -3\cos\left(\frac{4\pi}{25}t\right) + 7$$

$\therefore$ $a = 7$, $b = -3$, $n = \frac{4\pi}{25}$

Comment : The most common error was failing to realise that the time starts from low tide. Hence, b is negative.

Also, some students did not recognise the period of motion is $\frac{25}{2}$ (NOT 25).

(ii) Low tide today = 3 am

Low tide tomorrow = 3 am + 25 hrs = 4 am

Tide is rising fastest 3.125 hrs after low tide (see diagram above).

$\therefore$ Time of interest = 4 am + 3.125

= 7:07.30 am

Period is 12.5 hrs

$\therefore$ Latest time tomorrow = 7:07.30 + 12.5

= 7:37.30 pm

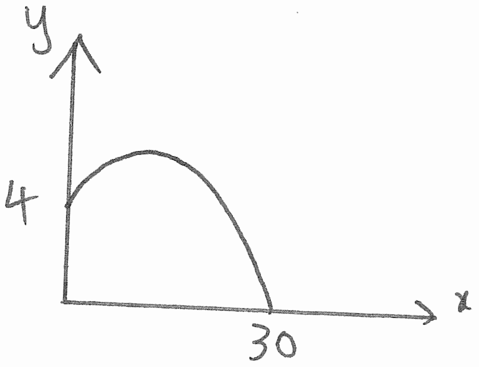
Question 13

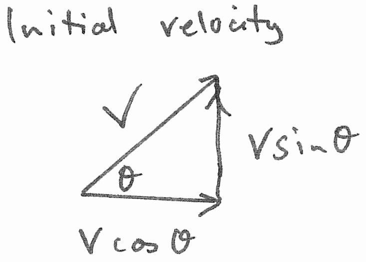
(a) (ii) Comment: Common error was failure to find the latest time tomorrow.

Many students used calculus and correctly identified times when $\frac{dr}{dt} = 0$ (to maximise r).

However, not all these times correspond to the tide rising. Students who used $b = +3$ in part (i) should be considering whether the "right" answer of 7:37 pm is mathematically correct according to their solution.

(b)





(i) $\ddot{y} = -10$ $\qquad \dot{y} = -10t + c$

when $t = 0$, $\dot{y} = V\sin\theta$ $\quad \therefore \; c = V\sin\theta$

$$\dot{y} = V\sin\theta - 10t$$

$$y = Vt\sin\theta - 5t^2 + c_1$$

when $t = 0$, $y = 4$ $\quad \therefore \; c_1 = 4$

$$y = Vt\sin\theta - 5t^2 + 4$$

Comment: This required showing where the equation y comes from. A clear indication of the initial conditions is expected.

Question 13

(b) (ii) $x = Vt\cos\theta \quad \therefore t = \dfrac{x}{V\cos\theta}$

$$y = V\sin\theta \times \frac{x}{V\cos\theta} - 5\times\frac{x^2}{V^2\cos^2\theta} + 4$$

$$= x\tan\theta - \frac{5x^2}{V^2}\sec^2\theta + 4$$

$$= x\tan\theta - \frac{5x^2}{V^2}(\tan^2\theta + 1) + 4$$

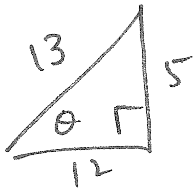
when $x = 30,\ y = 0$

$$0 = 30\times\frac{5}{12} - \frac{5\times30^2}{V^2}\left(\left(\frac{5}{12}\right)^2 + 1\right) + 4$$

$$\frac{4500}{V^2}\times\frac{169}{144} = \frac{33}{2}$$

$V^2 \doteqdot 320.076 \quad \therefore V \doteqdot 17.89$ m/s

Comment: Students prone to calculation errors in this question.



(iii) max. height when $\dot{y} = 0$

$-10t + V\sin\theta = 0 \quad \therefore t = \dfrac{V\sin\theta}{10}$

$$= \frac{17.89\times\frac{5}{13}}{10}$$

$t \doteqdot 0.688$ seconds

Question 13

(b) (iii) continued

when $t \doteqdot 0.688$, $y = 17.89 \times 0.688 \times \frac{5}{13} - 5 \times 0.688^2 + 4$

i.e. max height $\doteqdot 6.37$ m

Comment: One incorrect approach was to find the time of flight and halve it for the time at the peak. This does not work since the rock is thrown from a point above sea level.

Some incorrectly used $y = -4$ for the sea level.

(c) (i) $\angle CBD = \angle DCF$ ($\angle$ in alt. segment)

$\angle CAB = \angle DCF$ (alt. $\angle$s, $AB \parallel CE$)

$\therefore \angle CBD = \angle CAB$ (both equal $\angle DCF$)

(ii) In ΔCBD and ΔCAB

(i) $\angle CBD = \angle CAB$ (proven above)

(ii) $\angle BCD = \angle ACB$ (common $\angle$)

$\therefore \Delta CBD \mathrel{|||} \Delta CAB$ (equiangular)

$\frac{CB}{CD} = \frac{CA}{CB}$ (corr. sides of similar Δs in the same ratio)

$\therefore CB \times CB = AC \times DC$

i.e. $BC^2 = AC \times DC$

Comment: Some students need to improve their reasoning and justify all lines.

Question 14

(a) Prove $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 3.

Step 1 : Prove true for $n=1$.

$$n^3 + (n+1)^3 + (n+2)^3 = 1^3 + (1+1)^3 + (1+2)^3$$
$$= 1 + 8 + 27$$
$$= 36$$
$$= 3 \times 12$$

$\therefore$ Divisible by 3 for $n=1$.

Step 2 : Assume true for $n=k$.

i.e. assume $k^3 + (k+1)^3 + (k+2)^3 = 3A$ where A is some integer

Step 3 : Prove true for $n=k+1$.

i.e. prove $(k+1)^3 + (k+1+1)^3 + (k+1+2)^3 = 3B$ where B is some integer

$$LHS = (k+1)^3 + (k+2)^3 + (k+3)^3$$
$$= \underbrace{(k+1)^3 + (k+2)^3 + k^3} + 3 \times 3k^2 + 3 \times 9k + 27$$
$$= 3A + 9k^2 + 27k + 27 \quad \text{using the assumption}$$
$$= 3(A + 3k^2 + 9k + 9)$$
$$= 3B \quad \text{where } B = A + 3k^2 + 9k + 9 \text{ i.e. some integer}$$
$$= RHS$$

If true for $n=k$, proven true for $n=k+1$.

Since proven true for $n=1$, then by mathematical induction, proven true for all positive integers.

Question 14

(a) Comment: Some of the induction proofs need polishing – particularly the structure in Step 3. Most solutions used the assumption but there were errors in expanding $(k+3)^3$. The word "assume" in step 2 is essential.

(b) $\cos(2\cos^{-1}x) = 2\cos^2(\cos^{-1}x) - 1$

$= 2(\cos(\cos^{-1}x))^2 - 1$

$= 2x^2 - 1$ since $0 \le x \le 1$

$\therefore \int_0^{\frac{1}{2}} \cos(2\cos^{-1}x)\,dx = \int_0^{\frac{1}{2}} (2x^2-1)\,dx$

$= \left[\frac{2x^3}{3} - x\right]_0^{\frac{1}{2}}$

$= \frac{2}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{2} - 0$

$= -\frac{5}{12}$

Comment: Many students attempted to use substitution, letting $\theta = \cos^{-1}x$. Whilst some students successfully completed the question by this method, many did not and a common mistake was forgetting to consider

$\frac{d\theta}{dx}$.

Question 14

(c) (i)

$$m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$$

$$= \frac{a(p-q)(p+q)}{2a(p-q)}$$

$$= \frac{p+q}{2}$$

$\therefore$ PQ has the equation $y - ap^2 = \left(\frac{p+q}{2}\right)\left(x - 2ap\right)$.

Since PQ is a focal chord, it passes through (0, a).

$$\therefore \quad a - ap^2 = \frac{p+q}{2}\left(0 - 2ap\right)$$

$$a - ap^2 = -ap^2 - apq$$

$$a = -apq$$

$$\therefore pq = -1$$

Comment: Please explain where needed.

Comment: An alternative approach used the relationship $m_{PQ} = m_{PS} = m_{QS}$, in some form.

Another approach used by a few involved clearly stating the propety that $\overset{\text{for}}{\uparrow}$ a focal chord, the tangents at the endpoints meet at right angles on the directrix.

Question 14

(c) (ii) Normals at P and Q :

$$x+py = 2ap+ap^3 \quad ①$$

$$x+qy = 2aq+aq^3 \quad ②$$

Intersection point of normals :

① - ② $\quad y(p-q) = 2a(p-q)+a(p^3-q^3)$

$$y = 2a+a(p^2+pq+q^2)$$

$$= a(p^2+pq+q^2+2)$$

$$x = 2ap+ap^3 - pa(p^2+pq+q^2+2)$$

$$= -ap(pq+q^2)$$

$$= -apq(p+q)$$

Since $pq=-1$ $\qquad x = a(p+q)$

$$p+q = \frac{x}{a}$$

$$y = a(p^2+2pq+q^2-pq+2)$$

$$= a((p+q)^2+3)$$

$$y = a\left(\frac{x^2}{a^2}+3\right)$$

$$y = \frac{x^2}{a}+3a$$

Question 14

(c) (ii) continued

$$x^2 = a(y - 3a)$$

Vertex is at $(0, 3a)$

Focal length $= \frac{a}{4}$

$\therefore$ Focus at $\left(0, 3a + \frac{a}{4}\right)$ i.e. $\left(0, \frac{13a}{4}\right)$

Directrix is $y = 3a - \frac{a}{4}$ i.e. $y = \frac{11a}{4}$

Comment: Many students struggled to create the equation of the parabola. Of those students who did, some incorrectly interpreted the focal length as $\frac{1}{4}$ instead of $\frac{a}{4}$.

Many students wasted time on creating the equation of the normal (some also found the tangent as well), but you are able to write it using the Reference Sheet. The question did not ask you to derive the equation.

Question 14

(d) $P(x) = (x-a)^3 + (x-b)^2$

$P(b) = -8$ $\qquad (b-a)^3 + (b-b)^2 = -8$

$$(b-a)^3 = -8$$

$$b - a = -2$$

$P'(x) = 3(x-a)^2 + 2(x-b)$

If there are stationary points, then $P'(x) = 0$

$3(x-a)^2 + 2(x-b) = 0$

$3(x^2 - 2ax + a^2) + 2x - 2b = 0$

$3x^2 + x(2-6a) + (3a^2 - 2b) = 0$

Consider the discriminant

$$\left.\begin{aligned} \Delta &= (2-6a)^2 - 4\times3(3a^2-2b) \\ &= 4 - 24a + 36a^2 - 36a^2 + 24b \\ &= 24(b-a) + 4 \\ &= 24(-2) + 4 \end{aligned}\right\}$$

Comment: Some students got bogged down in algebra in this step.

$\Delta = -44$ Since $\Delta < 0$, $P'(x) = 0$ has no solutions.

$\therefore P(x)$ has no stationary points.

Comment: Solutions can be improved by making more clear the link between $P'(x) = 0$ having no solutions and $P(x)$ having no stationary points.